Slope Fields, Vector Fields, and Statistical Vector Fields for Longitudinal Data

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Abstract

Longitudinal data are often analyzed using a series of dynamic simultaneous equations. A *slope field* plots the expected change in one variable with respect to another variable for ranges of each of those variables. A *vector field* plots the expected changes in variables with respect to each other and with respect to time. A *statistical slope fields* or a *statistical vector field* offer a similar display, but are exploratory graphs generated from raw data and include information about the variations due to sampling.

1 Slope Fields

One may think of a slope field as a scatterplot where at each selected pair (x, y) denoting a location with respect to abscissa and ordinate axes is plotted the expected change in y for a one unit change in x as indicated by a short line segment of that slope. As an example, suppose a variable y were related to its slope with respect to time t such that

$$\Delta y(t)/\Delta t = \beta(y(t) - C) \tag{1}$$

where $y(t)$ is some time varying score, $\Delta y(t)/\Delta t$ is the expected change in $y(t)$ over some fixed interval of time Δt, β is some fixed coefficient, and C is some asymptotic value (i.e. *fixed point equilibrium*). From this equation, a slope field may be plotted such that on each point in a grid of pairs of values of Y and t a line segment is centered with the expected slope given that pair of values (i.e. *initial conditions*). The lengths of the line segments are all equal to one another. Three salient characteristics of the slope field shown are that (a) since each row of line segments has no variance in slope, the slopes are independent of time, (b) since each column of line segments has variance in slope, the slope is not independent of y, and (c) since the slopes are near zero when y is near 60, the equilibrium C in this plot must be near 60.

Slope fields may be plotted in which one variable represents time, or may also be plotted so as to visualize the expected slope of two variables with respect to each other. The expected slope could potentially vary with respect to the value of one, both or neither of the two variables.

*Funding for this work was provided in part by NIH Grants AG–14983 and AG–07137. Correspondence may be addressed to Steven M. Boker, Department of Psychology, The University of Notre Dame, Notre Dame Indiana 46556, USA; email sent to sboker@nd.edu. Vector field and slope field plots can be created in Matlab, Mathematica, Maple and Splus or R. Example code to produce the figures in this entry may be obtained from http://www.nd.edu/˜ sboker.

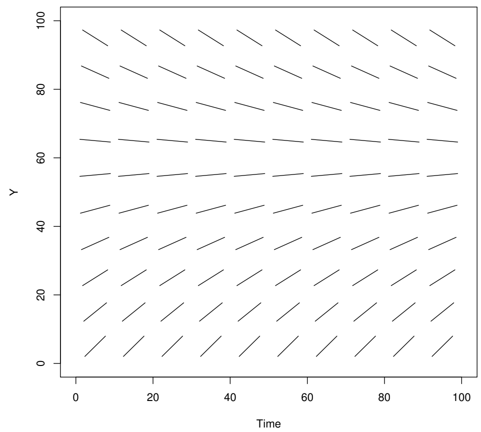

Figure 1: Slope field plotted for Equation 1.

2 Vector Fields

Vector fields differ from slope fields in that they are directional, implying some evolution of the slope (i.e. first derivative) with respect to time. Vector fields are useful for visualizing the implications of differential equations over a range of initial conditions [5, 10]. A vector field is composed of a grid of arrows that may vary in direction and length. Direction and length may be mapped to a variety of concepts. The most commonly used vector field display maps the vectors so that given the values of the variables in a system at time t plotted as the base of a vector, the length and the direction of the vector point to the values of those variables after some chosen interval τ has elapsed.

As an example in continuous time, the relationship between two variables X and Y might be modeled as a set of simultaneous differential equations

$$\begin{aligned} dX/dt &= \alpha_x + \beta_x X(t) + \gamma_x Y(t) \\ dY/dt &= \alpha_y + \beta_y Y(t) + \gamma_y X(t) \end{aligned} \tag{2}$$

where the coefficients (α, β, γ) for each variable are used to describe the instantaneous trajectory of the bivariate processes. Or equivalently in discrete time might be formed as a set of simultaneous difference equations

$$\begin{aligned} \Delta X(t)/\Delta t \quad & x \quad \alpha_x + \beta_x X(t-\Delta t) + \gamma_x Y(t-\Delta t) \\ \Delta Y(t)/\Delta t \quad & x \quad \alpha_y + \beta_y Y(t-\Delta t) + \gamma_y X(t-\Delta t) \end{aligned} \tag{3}$$

where Δt is defined by the application, and the coefficients for each variable describe the step-by-step trajectory of the bivariate processes [4, 7, 8]. Such a system is plotted in Figures 2–a and –b.

Vector field plots such as that shown in Figure 2–b can be read to provide several forms of information. In areas of the graph where vector lengths are small, the system is near an equilibrium. If vectors point away from an equilibrium, then that equilibrium is unstable. If vectors point towards an equilibrium, then that equilibrium is stable. If vectors appear to "circle" an equilibrium, then the system may oscillate under some initial conditions.

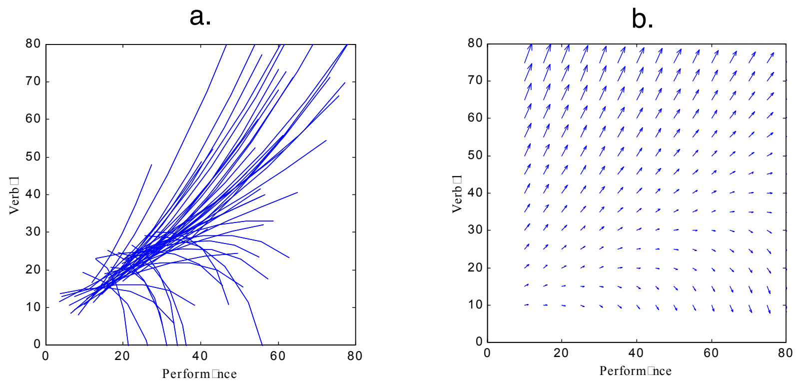

Figure 2: Trajectories and vector field generated from coefficents estimated from WISC data from $N = 204$ Children aged 6–11 [6]. (a) Hypothetical individual trajectories evolving over a lifetime from a few selected initial conditions. (b) A vector field plotting the evolution of a grid of initial conditions over a short interval of time.

Vector fields are also used to visualize the relationship between a variable and its first and second derivatives with respect to time [1, 3]. The vector field in Figure 3 plots the expected change in x and its first derivative $\dot{x}$ over a short interval of time for the damped linear oscillator system

$$\ddot{x} = \eta x + \zeta \dot{x} \tag{4}$$

when η and ζ are both negative. This system has a stable equilibrium at $x = 0$ and oscillates about that equilibrium for a time that is dependent on its initial conditions.

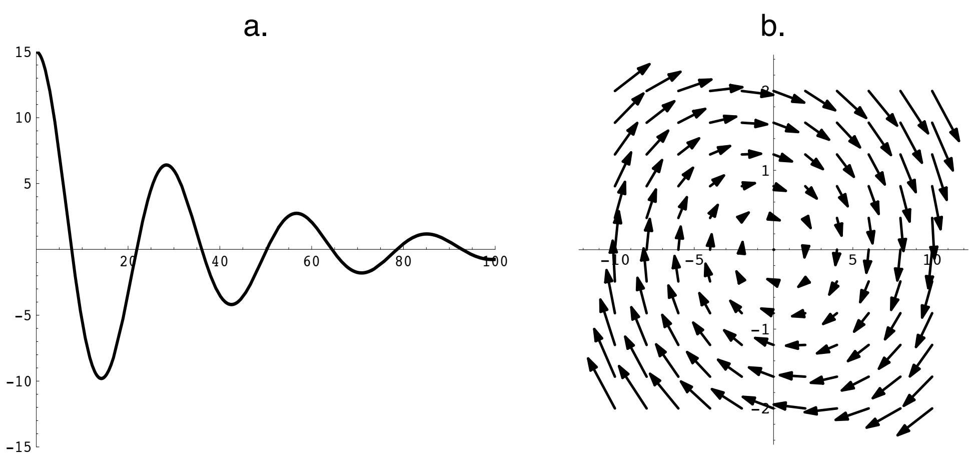

Figure 3: (a) Time series plot and (b) vector field plot of a damped linear oscillator conforming to Equation 4.

3 Statistical Slope Fields

When performing exploratory analyses on longitudinal data, it is often useful to visualize the expected change of one variable with respect to another or in a variable with respect to time. Statistical slope fields employ nonparametric methods of local aggregation or smoothing to develop local estimates of the derivatives of systems and then plot these empirically derived estimates. Typically, a statistical slope field appears similar to a parametric slope field, but varies the length of the line segment as a means of displaying the relative proportion of the data near to the center of the line segment.

As an example, suppose a random sample of 100 individuals of different ages were drawn from a population whose scores were evolving according to the autoregressive system from Equation 1. Suppose each of these individuals were measured at two occasions separated by 15 years and that the measured score were the sum of a true score and normally distributed independent error. These longitudinal scores could be plotted as in Figure 4–a. The statistical slope field of the same data was calculated using loess smoothing to locally estimate the derivative of the score with respect to time at the center of each point in a grid of age and score pairs. In some areas there is no data and thus no estimate is made. Derivative estimates made larger Ns are plotted as longer line segments.

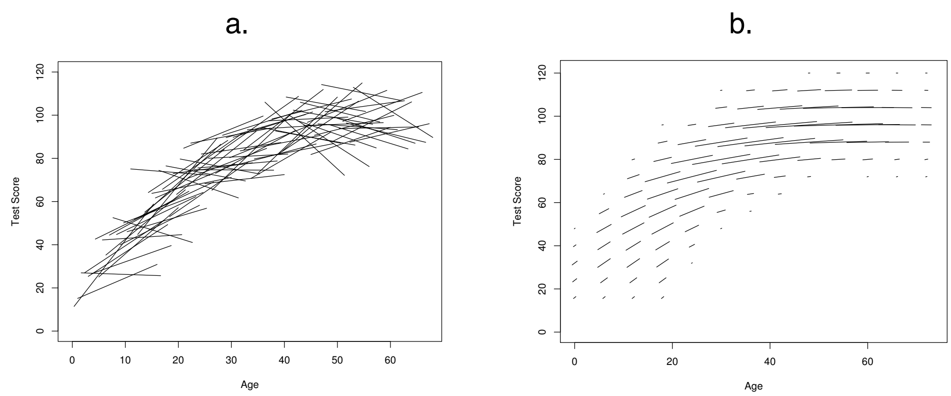

Figure 4: (a) Longitudinal time series plot and (b) statistical slope field plot of an autoregressive process conforming to Equation 1.

4 Statistical Vector Fields

A statistical vector field [2] is similar to a statistical slope field except that there is a time direction of the evolution of the system and an estimate of the variability of the slopes is made in the locality of each initial condition pair. The direction of the vector represents the expected change in the value of the variable shown on the ordinate axis with respect to a unit change in the variable shown on the abscissa. The length of the vector plots the proportion of the data in the vicinity of the initial condition pair at the base of the vector. The estimated standard deviation of this slope is plotted as a gray arc centered around the vector. An example statistical vector field of data conforming to Equation 1 is plotted in Figure 5 [9, 8].

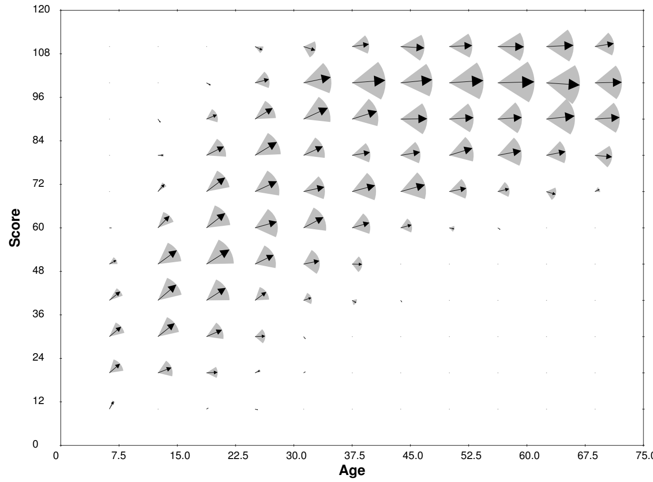

Figure 5: A statistical vector field plot of data from an autoregressive process conforming to Equation 1.

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