Latent Differential Equations

Steven M. Boker

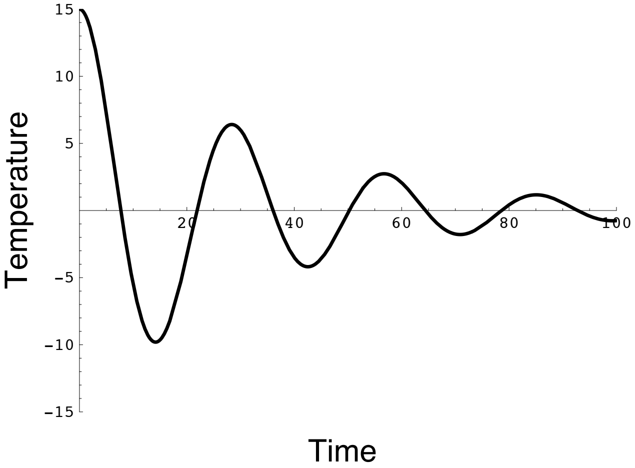
Department of Psychology
University of Virginia

Dynamical Systems Analysis Workshop Part 4

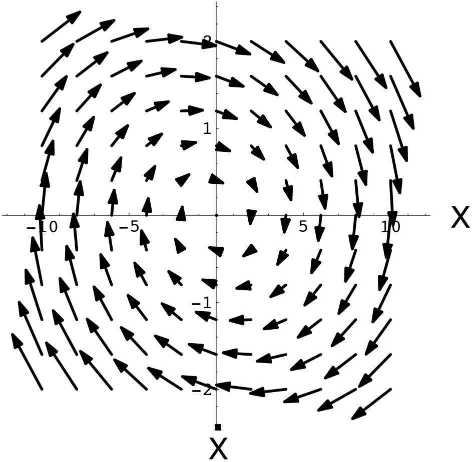
Modern Modeling Methods
May 22, 2017

Overview

- Introduction to dynamical systems.
- Univariate first order LDE.
- Univariate second order LDE.



Temperature Trajectory Versus Time



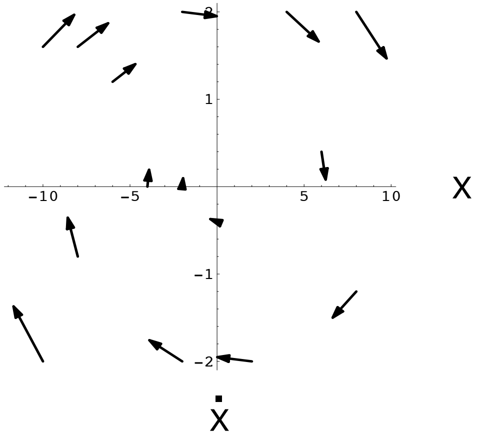
Difference from equilibrium and the first and second derivative of temperature

Independent Samples ($\Delta t = 1$)

Suppose we draw N independent longitudinal samples (measurement bursts) from an individual and construct a data matrix as follows.

$$X^{(5)} = \begin{bmatrix} x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & x_{(1,4)} & x_{(1,5)} \\ x_{(2,1)} & x_{(2,2)} & x_{(2,3)} & x_{(2,4)} & x_{(2,5)} \\ x_{(3,1)} & x_{(3,2)} & x_{(3,3)} & x_{(3,4)} & x_{(3,5)} \\ x_{(4,1)} & x_{(4,2)} & x_{(4,3)} & x_{(4,4)} & x_{(4,5)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(N,1)} & x_{(N,2)} & x_{(N,3)} & x_{(N,4)} & x_{(N,5)} \end{bmatrix}.$$

This results in $X^{(5)}$ being an $N \times 5$ data matrix whose columns span a time of 4 units.



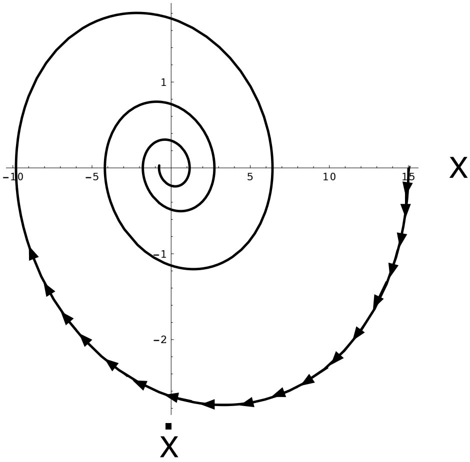
A sample of independent ordered bursts from the vector field

Time Delay Embedding ($\tau = 1$, $\Delta t = 1$)

Now suppose we measure the individual for P successive occasions and construct a data matrix as follows.

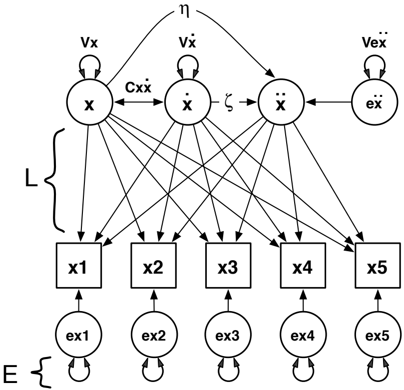
$$X^{(5)} = \begin{bmatrix} x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & x_{(1,4)} & x_{(1,5)} \\ x_{(1,2)} & x_{(1,3)} & x_{(1,4)} & x_{(1,5)} & x_{(1,6)} \\ x_{(1,3)} & x_{(1,4)} & x_{(1,5)} & x_{(1,6)} & x_{(1,7)} \\ x_{(1,4)} & x_{(1,5)} & x_{(1,6)} & x_{(1,7)} & x_{(1,8)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(1,P-4)} & x_{(1,P-3)} & x_{(1,P-2)} & x_{(1,P-1)} & x_{(1,P)} \end{bmatrix}.$$

If $P = N + 8$ this results in $X^{(5)}$ being an $N \times 5$ data matrix whose columns span a time of 4 units.



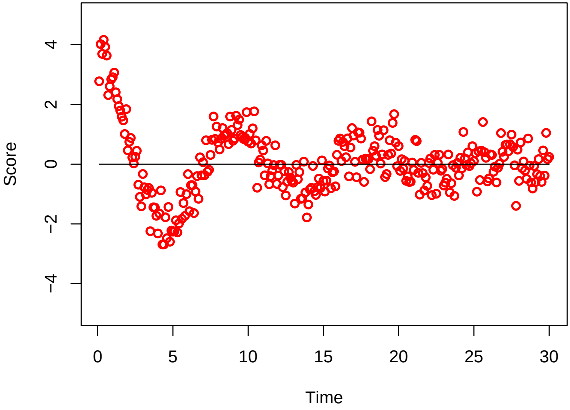
Time delay embedding from a single trajectory.

Latent Differential Equations (LDE)



Univariate second order LDE path model

Univariate Time Series



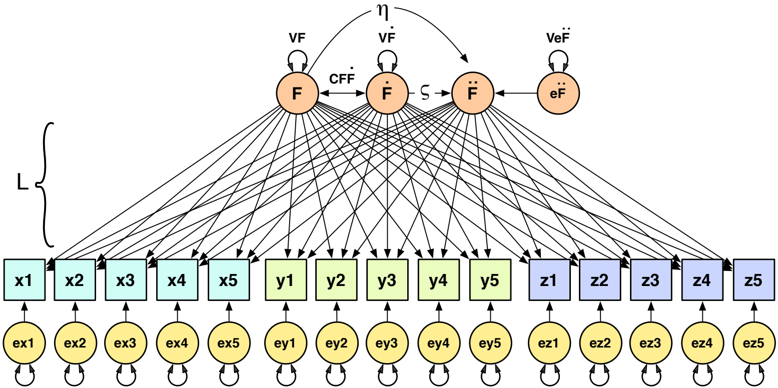
Univariate LDE Model Matrices

$$
\begin{aligned}
\mathbf{L} &= \begin{bmatrix} 1 & -2\tau\Delta t & (-2\tau\Delta t)^2/2 \\ 1 & -1\tau\Delta t & (-1\tau\Delta t)^2/2 \\ 1 & 0 & 0 \\ 1 & 1\tau\Delta t & (1\tau\Delta t)^2/2 \\ 1 & 2\tau\Delta t & (2\tau\Delta t)^2/2 \end{bmatrix} \\
\mathbf{A} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & \zeta & 0 \end{bmatrix} \\
\mathbf{S} &= \begin{bmatrix} V_F & C_{FdF} & 0 \\ C_{FdF} & V_{dF} & 0 \\ 0 & 0 & V_{d2F} \end{bmatrix}
\end{aligned}
$$

$$\mathbf{R} = \mathbf{L}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}(\mathbf{I} - \mathbf{A})^{-1'}\mathbf{L}' + \mathbf{U}^2$$

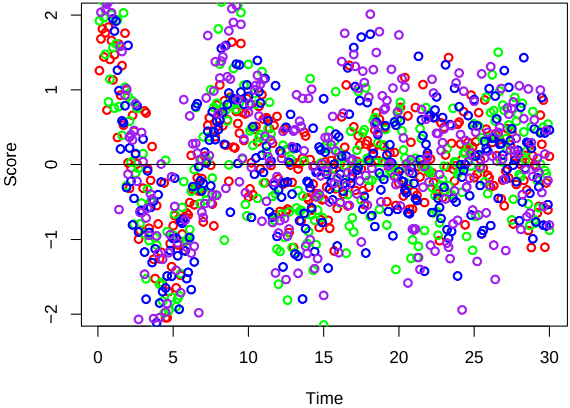
Univariate LDE FIML Model

Go to LDEUnivariateFIMLExample.R



Multivariate Latent Differential Equation Structural Model

Multivariate Single Factor Time Series



Multivariate Time Delay Embedding

$$X^{(5)} = \begin{bmatrix} x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & x_{(1,4)} & x_{(1,5)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(1,P-4)} & x_{(1,P-3)} & x_{(1,P-2)} & x_{(1,P-1)} & x_{(1,P)} \end{bmatrix}$$

$$Y^{(5)} = \begin{bmatrix} y_{(1,1)} & y_{(1,2)} & y_{(1,3)} & y_{(1,4)} & y_{(1,5)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{(1,P-4)} & y_{(1,P-3)} & y_{(1,P-2)} & y_{(1,P-1)} & y_{(1,P)} \end{bmatrix}$$

$$Z^{(5)} = \begin{bmatrix} z_{(1,1)} & z_{(1,2)} & z_{(1,3)} & z_{(1,4)} & z_{(1,5)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{(1,P-4)} & z_{(1,P-3)} & z_{(1,P-2)} & z_{(1,P-1)} & z_{(1,P)} \end{bmatrix}$$

$$W^{(5)} = \begin{bmatrix} w_{(1,1)} & w_{(1,2)} & w_{(1,3)} & w_{(1,4)} & w_{(1,5)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{(1,P-4)} & w_{(1,P-3)} & w_{(1,P-2)} & w_{(1,P-1)} & w_{(1,P)} \end{bmatrix}$$

Now, we can rbind(X,Y,Z,W) (i.e., augment) these matrices into a $(P-4) \times 20$ data matrix $D^5 = X|Y|Z|W$ that can be fit by the LDE model.

Multivariate LDE Model Matrices

$$\mathbf{L} = \begin{bmatrix} 1 & -2\tau\Delta t & (-2\tau\Delta t)^2/2 \\ 1 & -1\tau\Delta t & (-1\tau\Delta t)^2/2 \\ 1 & 0 & 0 \\ 1 & 1\tau\Delta t & (1\tau\Delta t)^2/2 \\ 1 & 2\tau\Delta t & (2\tau\Delta t)^2/2 \\ a & -2a\tau\Delta t & a(-2\tau\Delta t)^2/2 \\ a & -1a\tau\Delta t & a(-1\tau\Delta t)^2/2 \\ a & 0 & 0 \\ a & 1a\tau\Delta t & a(1\tau\Delta t)^2/2 \\ a & 2a\tau\Delta t & a(2\tau\Delta t)^2/2 \\ b & -2b\tau\Delta t & b(-2\tau\Delta t)^2/2 \\ b & -1b\tau\Delta t & b(-1\tau\Delta t)^2/2 \\ b & 0 & 0 \\ b & 1b\tau\Delta t & b(1\tau\Delta t)^2/2 \\ b & 2b\tau\Delta t & b(2\tau\Delta t)^2/2 \end{bmatrix}$$

Multivariate LDE Model Matrices

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & \zeta & 0 \end{bmatrix} \\
\mathbf{S} &= \begin{bmatrix} V_F & C_{FdF} & 0 \\ C_{FdF} & V_{dF} & 0 \\ 0 & 0 & V_{d2F} \end{bmatrix}
\end{aligned}
$$

$$\mathbf{R} = \mathbf{L}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{S}(\mathbf{I} - \mathbf{A})^{-1'}\mathbf{L}' + \mathbf{U}^2$$

Multivariate LDE FIML Example

Go to LDEMultivariateFIMLExample.R