Introduction to Dynamical Systems Analysis

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Dynamical Systems Analysis Workshop Part 1

Modern Modeling Methods
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What are Dynamical Systems?

- **System** — a set of P variables that change over time, and that have some natural connectedness.
- **State space** — an P-dimensional space composed of all of the values of all of the variables that a system takes on over an extended period of time.
- **Dynamical system** — a system in which the present state of the system (its position in state space) is somehow dependent on previous states of the system (previous positions in state space).
- **Initial conditions** — a selected point in a state space.
- **Trajectory in state space** — a set of states in a state space that evolves over time from initial conditions. A single instantiation of a dynamical system.

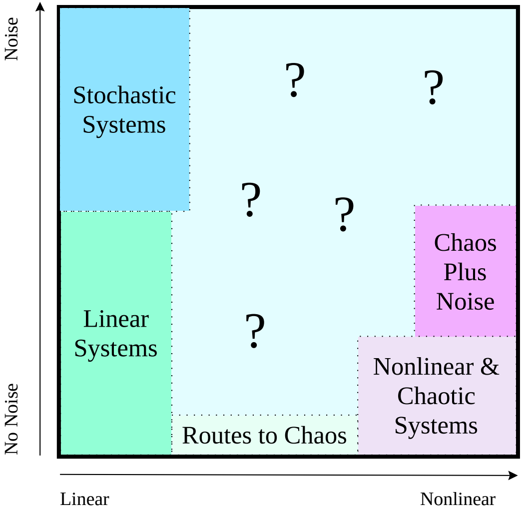
What are Linear Systems?

- **Time derivative** — the instantaneous change in a variable with respect to time.
- **Phase space** — a state space that includes time derivatives.
- **Deterministic system** — a system in which the present state is *entirely* dependent on previous states of the system.
- **Linear system** — a system in which all of the dependence of the current state on previous states can be expressed in terms of a linear combination.
- **Linear stochastic system** a linear system in which the residual unpredictable portions can be expressed as additive, independent, identically distributed, random variables.

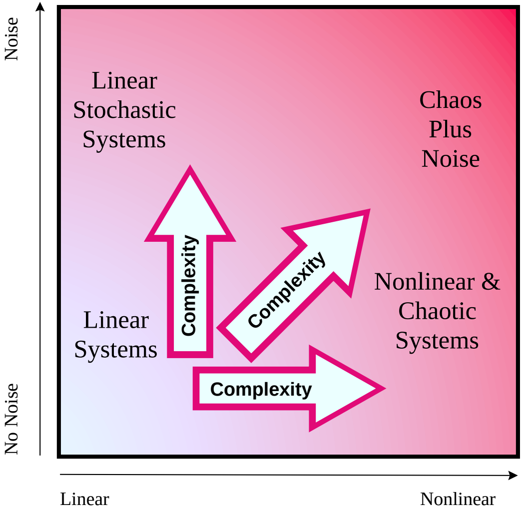
What are Nonlinear Systems?

- **Nonlinear system** — a system in which the dependence of the current state on previous states cannot be expressed entirely as a linear combination.
- **Sensitive dependence on initial conditions** — An infinitesimal difference in two initial states of a dynamical system causes an exponential divergence between the resulting trajectories.
- **Dissipative system** — a system whose state space is bounded.
- **Chaotic system** — a dissipative dynamical system which exhibits sensitive dependence on initial conditions.
  - A chaotic system is always nonlinear.
  - A nonlinear system is not necessarily chaotic.

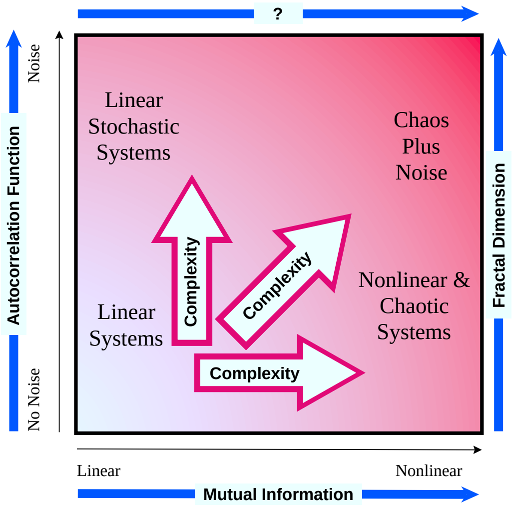
Types of Dynamical Systems



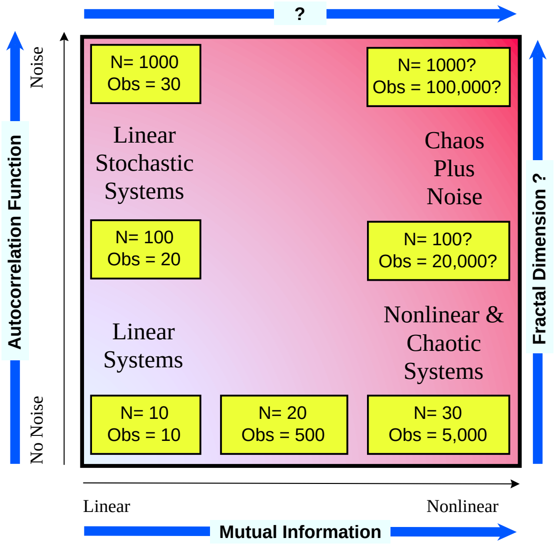
Complexity


Noise
No Noise
Linear Stochastic Systems
Chaos Plus Noise
Complexity
Complexity
Linear Systems
Nonlinear & Chaotic Systems
Complexity
Linear
Nonlinear

Measures of Complexity



Data Constraints



Some Examples of Dynamical Systems

- A pendulum.
- The movement of the Earth.
- The pumping of the heart.
- Ovarian hormone cycles.
- Postural stability.
- Interpersonal coordination in dance or conversation.
- Daily affect.
- Grief.
- Depression or bipolar disorder.
- Learning and cognitive plasticity.
- Developmental change.

Sampling Issues

- A system may be changing continuously over time, but we can only measure it discretely.
- Example: a strobe on a pendulum (The Pendulum Disco).
- We don't necessarily know what is happening between measurements.
  - Continuity is often assumed.
  - A system may exhibit sudden jumps and still be modeled as a continuous system.
- Assuming linearity for a truly linear system is fine as long as the samples are close enough together to capture the dynamical behavior of the system.
- Assuming linearity for a truly nonlinear system could cause problems.

Time–Delay Embedding

- Suppose we have a time series,

$$X = \{x_1, x_2, x_3, \ldots, x_P\}.$$

- A four dimensional *time–delay embedded state space*, $\mathbf{X}^{(4)}$, for the time series can be constructed as

$$\mathbf{X}^{(4)} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \\ x_3 & x_4 & x_5 & x_6 \\ \vdots & \vdots & \vdots & \vdots \\ x_{P-3} & x_{P-2} & x_{P-1} & x_P \end{bmatrix} .$$

Time–Delay Embedding

- Due to theorems from Whitney and later Takens a state space can capture all of the dynamics in the time series *if*
  1. the τ is chosen properly, and
  2. the number of dimensions of the state space is sufficient.

Time–Delay Embedding for a State Space Plot

- Suppose we have a time series,

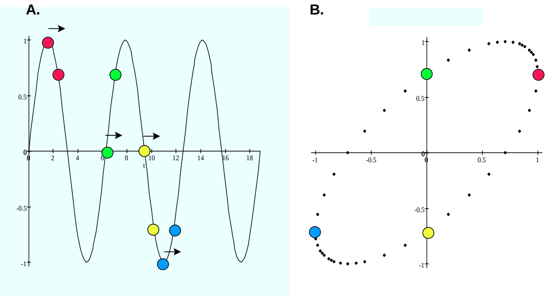
$$X = \{x_1, x_2, x_3, \ldots, x_P\}.$$

- We can choose a lag, τ and construct a two dimensional time–delay embedded state space in order to make a state space plot:

$$\mathbf{X}^{(2)} = \begin{bmatrix} x_1 & x_{1+\tau} \\ x_2 & x_{2+\tau} \\ x_3 & x_{3+\tau} \\ \vdots & \vdots \\ x_{P-\tau} & x_P \end{bmatrix}.$$

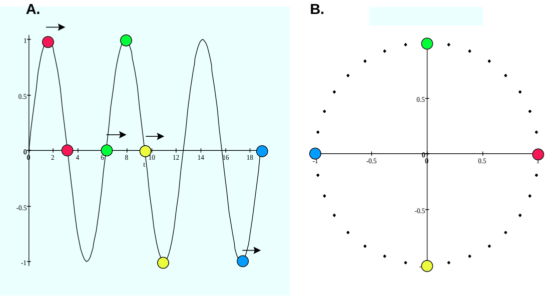
- We do this in order to illustrate how the time interval between measurements can affect the autocorrelation you calculate from a time series.

Time–Delay Embedded State Space and Lag



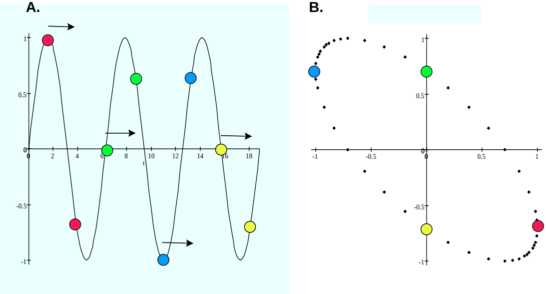
(A) Time series plot of y_t.
(B) State space plot of $y_{(t)}$ and $y_{(t+\pi/4)}$.

Time–Delay Embedded State Space and Lag



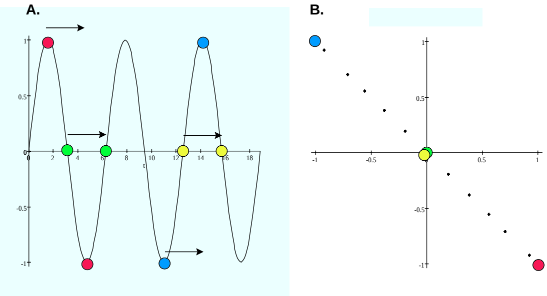
(A) Time series plot of y_t.
(B) State space plot of $y_{(t)}$ and $y_{(t+\pi/2)}$.

Time–Delay Embedded State Space and Lag



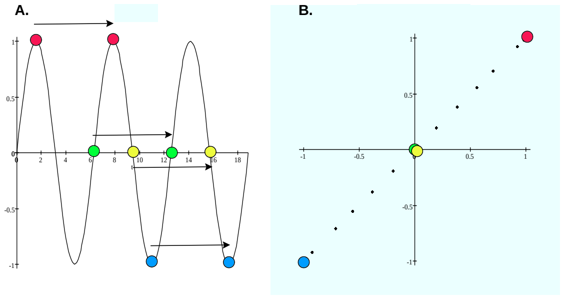
(A) Time series plot of y_t.
(B) State space plot of $y_{(t)}$ and $y_{(t+3\pi/4)}$.

Time–Delay Embedded State Space and Lag



(A) Time series plot of y_t.
(B) State space plot of $y_{(t)}$ and $y_{(t+\pi)}$.

Time–Delay Embedded State Space and Lag

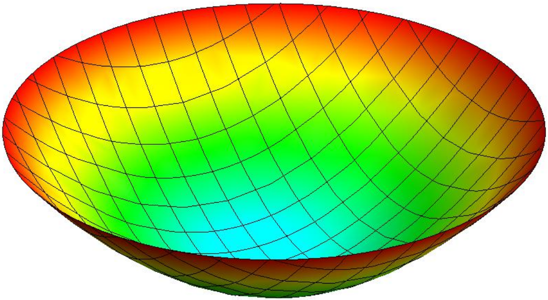


(A) Time series plot of y_t.
(B) State space plot of $y_{(t)}$ and $y_{(t+\pi)}$.

Attractors

- Attractors come in a variety of shapes, sizes, dimensions and types.
  - Point Attractor.
  - Point Repellor.
  - Limit Cycle.
  - Saddle.
  - Cusp.
  - Strange Attractor or Chaotic Attractor.
- Attractors give an idea of the underlying geometry of a dynamical system.
- One can examine the attractor in a set of data without having to assume that the underlying dynamical system is linear or nonlinear, periodic or aperiodic, noisy or chaotic.

Surface of a Point Attractor



What Do the Data Look Like?

- Data must be repeated observations.
- Data may be multivariate or univariate.
- We expect that the ordering of the observations matters.
- The easiest data are equal interval time series.

Prepare to Run the Workshop Scripts

If you are having trouble, raise your hand and a TA will assist you

1. If you do not yet have R, download it now.
2. If you have a Mac and OS X 10.11 (El Capitan) or earlier, download R 3.3.
3. Be sure the following libraries (and dependencies) are installed
   3.1 Lattice
   3.2 deSolve
   3.3 psych
   3.4 nlme
   3.5 stats
   3.6 OpenMx
4. Create a folder (i.e., directory) for the workshop and copy the workshop files onto it.

Boker, S. M. (2002). Consequences of continuity: The hunt for intrinsic properties within parameters of dynamics in psychological processes. *Multivariate Behavioral Research*, *37*(3), 405–422.

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