Multidimensional Reliability: A Proposal with Examples

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- Interest in congeneric measurement models (linear latent variable models with normal distribution theory).

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- Reliability will be expressed as McDonald's ω.
- Goal here is to find an expression for the reliabilities of individual factors.

Review

- The congeneric model for a p-item test:

$$\underset{p\times 1}{\mathbf{y}} = \underset{p\times 1}{\boldsymbol{\nu}} + \underset{p\times m}{\boldsymbol{\Lambda}}\underset{m\times 1}{\boldsymbol{\eta}} + \underset{p\times 1}{\boldsymbol{\epsilon}};$$

$$\boldsymbol{\eta} \sim \mathrm{N}(\mathbf{0}, \underset{m\times m}{\boldsymbol{\Psi}}); \quad \text{and} \quad \boldsymbol{\epsilon} \sim \mathrm{N}(\mathbf{0}, \underset{p\times p}{\boldsymbol{\Theta}}).$$

- Consider the unit-weighted sum-score $Y = \mathbf{1}'\mathbf{y}$.
- $\mathrm{var}(Y) = \mathbf{1}'(\boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta})\mathbf{1}$
- Multidimensional reliability is given by McDonald's

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)}$$

- Interested in cases where $m > 1$.

Source for Examples

- AED: Alcoholic Energy Drink Expectancies Scale (Miller et al., In press).
- $p = 15$ Likert items. Range: 1–6.
- $N = 3064$

- Maximum Likelihood Estimation:
  - **lavaan** (Rosseel, 2012) in R
  - Mplus (Muthén & Muthén, 2017).
- MLE:

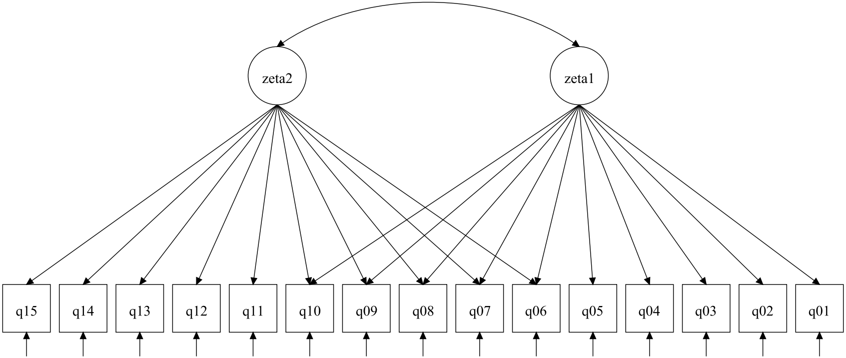
$$\hat{\omega} = \omega(\hat{\mathbf{\Lambda}}, \hat{\mathbf{\Psi}}, \hat{\mathbf{\Theta}}).$$

Two-Dimensional Scale

To fix ideas I begin with

- Two-factor measurement model
- 5 of 15 items cross-load on both factors
- Factors are correlated

Two-Dimensional Model



Two-dimensional Reliability

$$\omega = \frac{\mathbf{1}' \left[\lambda_1 \psi_{11} \lambda_1' + \lambda_1 \psi_{12} \lambda_2' + \lambda_2 \psi_{21} \lambda_1' + \lambda_2 \psi_{22} \lambda_2'\right] \mathbf{1}}{\text{var}\, Y}$$
$$= 0.935$$

Define subscale reliabilities as

$$\omega_1 = \frac{\mathbf{1}' \left[\lambda_1 \psi_{11} \lambda_1' + \lambda_1 \psi_{12} \lambda_2'\right] \mathbf{1}}{\text{var}(Y)} = 0.266$$
$$\omega_2 = \frac{\mathbf{1}' \left[\lambda_2 \psi_{22} \lambda_2' + \lambda_2 \psi_{21} \lambda_1'\right] \mathbf{1}}{\text{var}(Y)} = 0.670$$

So that

$$\omega = \omega_1 + \omega_2.$$

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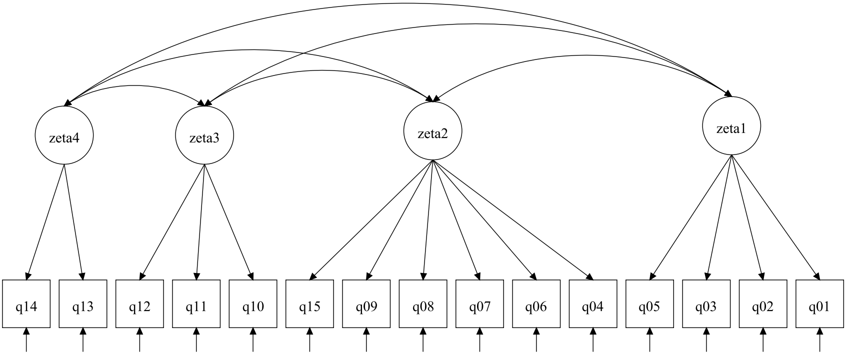
Multidimensional Scale

Now consider a 4-factor scale.

- 4-factor measurement model
- Items form a perfect cluster configuration
- Factors are correlated

Multidimensional Model



$$\omega = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = 0.954$$

Define the (general) subscale reliability for factor k as

$$\omega_k = \frac{\mathbf{1}'\boldsymbol{\lambda}_k\boldsymbol{\psi}'_k\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)}.$$

Then for this 4-dimensional model

$$\omega_1 = \frac{\mathbf{1}'\boldsymbol{\lambda}_1\boldsymbol{\psi}'_1\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = .307; \quad \text{and} \quad \omega_2 = \frac{\mathbf{1}'\boldsymbol{\lambda}_2\boldsymbol{\psi}'_2\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = .374;$$

$$\omega_3 = \frac{\mathbf{1}'\boldsymbol{\lambda}_3\boldsymbol{\psi}'_3\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = .209; \quad \text{and} \quad \omega_4 = \frac{\mathbf{1}'\boldsymbol{\lambda}_4\boldsymbol{\psi}'_4\mathbf{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = .063.$$

Detail on Subscale Reliability

The subscale reliability for factor k as can be further explicated as

$$\omega_k = \frac{\mathbf{1}'\lambda_k \boldsymbol{\psi}_k' \Lambda' \mathbf{1}}{\text{var}(Y)} = \frac{\mathbf{1}'\lambda_k \psi_{kk} \lambda_k' \mathbf{1} + \sum_{j \neq k} \mathbf{1}'\lambda_k \psi_{kj} \lambda_j' \mathbf{1}}{\text{var}(Y)}$$
$$= \text{standard reliability} + \text{sum of all "cross-reliabilities"}.$$

- If factor k is uncorrelated with the other factors, then ω_k reduces to standard reliability.
- Otherwise, ω_k incorporates all the correlations between the loadings on factor k and the loadings on the remaining factors.

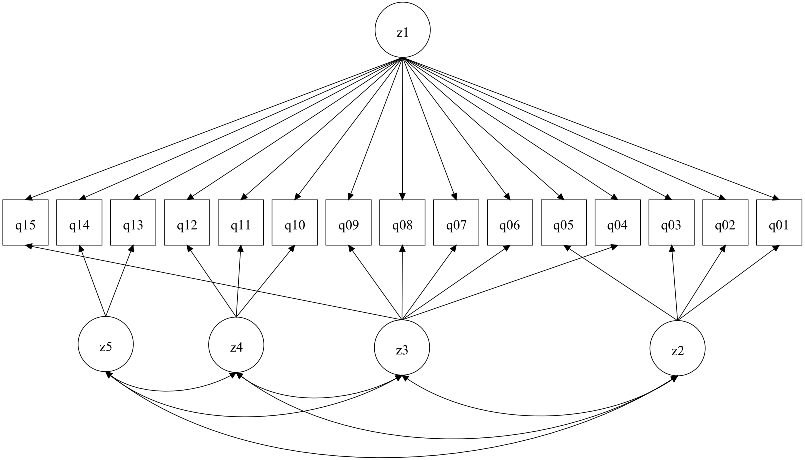
Hierarchical or Bifactor Scale

Now let's consider the hierarchical or bifactor model.

- One general factor for all 15 items
- 4 correlated specific factors
- General factor is uncorrelated with specific factors
- Items form a perfect cluster configuration on each specific factor

This example also demonstrates that a the reliability of a subset of factors, as opposed that for a single factor, can also be obtained.

Bifactor Model



Bifactor Reliability

The multidimensional reliability is

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = .957$$

The subscale reliability for the general factor is

$$\omega_1 = \frac{\mathbf{1}'\boldsymbol{\lambda}_1\boldsymbol{\psi}_1'\boldsymbol{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = \frac{\mathbf{1}'\boldsymbol{\lambda}_1\psi_{11}\boldsymbol{\lambda}_1'\mathbf{1}}{\mathrm{var}(Y)} = .679.$$

The multidimensional subscale reliability for the specific factors is

$$\omega_{2:5} = \frac{\mathbf{1}'\boldsymbol{\Lambda}_{2:5}\boldsymbol{\Psi}_{2:5,2:5}\boldsymbol{\Lambda}_{2:5}'\mathbf{1}}{\mathrm{var}(Y)} = .278.$$

The specific subscale reliabilities may be obtained as in the multidimensional case.

Extended Congeneric Model

The extended congeneric measurement model is required for higher-order models. The extended model is

$$\mathbf{y} = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\epsilon} \quad \text{with} \quad \boldsymbol{\eta} = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta}.$$

Thus, $\text{var}(Y) = \mathbf{1}' \left\{ \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\Psi}[(\mathbf{I} - \mathbf{B})^{-1}]'\boldsymbol{\Lambda}' + \boldsymbol{\Theta} \right\} \mathbf{1}.$

Bentler's extension to ω is

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\Psi}[(\mathbf{I} - \mathbf{B})^{-1}]'\boldsymbol{\Lambda}'\mathbf{1}}{\text{var}(Y)}.$$

The MLE estimate is now

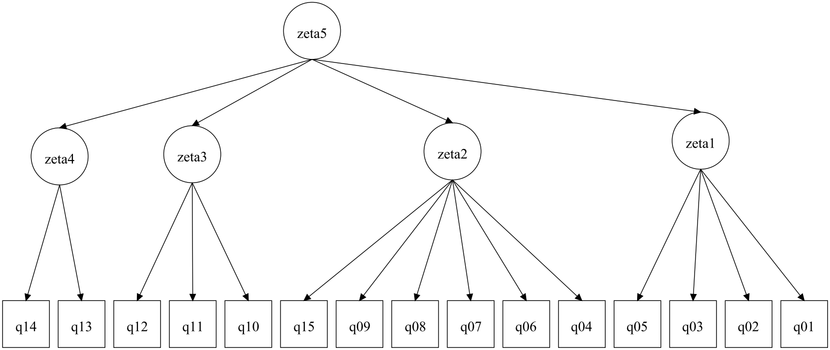
$$\hat{\omega} = \omega(\hat{\boldsymbol{\Lambda}}, \hat{\mathbf{B}}, \hat{\boldsymbol{\Psi}}, \hat{\boldsymbol{\Theta}}).$$

Second-order Measurement Model

Now consider a second-order measurement model.

- Items form a perfect cluster configuration on each of 4 factors
- A second-order factor accounts for the correlation among the 4 first-order factors.
- “Direct” reliability among first-order factors
- ”Indirect” reliability for second-order factor

2nd Order Measurement Model



2nd-order Multidimensional Reliability

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\Psi}(\mathbf{I}-\mathbf{B})^{-1}]'\boldsymbol{\Lambda}'\mathbf{1}}{\mathrm{var}(Y)} = 0.954$$

Let $\tilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}$. Define subscale reliability for factor k as

$$\omega_k = \frac{\mathbf{1}'\tilde{\boldsymbol{\lambda}}_k \boldsymbol{\psi}'_k \tilde{\boldsymbol{\Lambda}}'\mathbf{1}}{\mathrm{var}(Y)}.$$

Then

$$\omega_1 = .028; \quad \omega_2 = .039; \quad \omega_3 = .008; \quad \omega_4 = .025;$$
$$\omega_5 = .854.$$

2nd-order Multidimensional Reliability, cont'd

Because of the presence of $(\mathbf{I}-\mathbf{B})^{-1}$ in the expression, the interpretation of the subscale reliabilities may be obscure. However, in this case,

$$(\mathbf{I}-\mathbf{B})^{-1} = \mathbf{I}+\mathbf{B}.$$

Thus

$$\omega = \frac{\mathbf{1}'\mathbf{\Lambda}(\mathbf{I}+\mathbf{B})\mathbf{\Psi}(\mathbf{I}+\mathbf{B})'\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = 0.954$$

Define the layer reliabilities as

$$\Omega_1 = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = \omega_1+\omega_2+\omega_3+\omega_4 = .100;$$

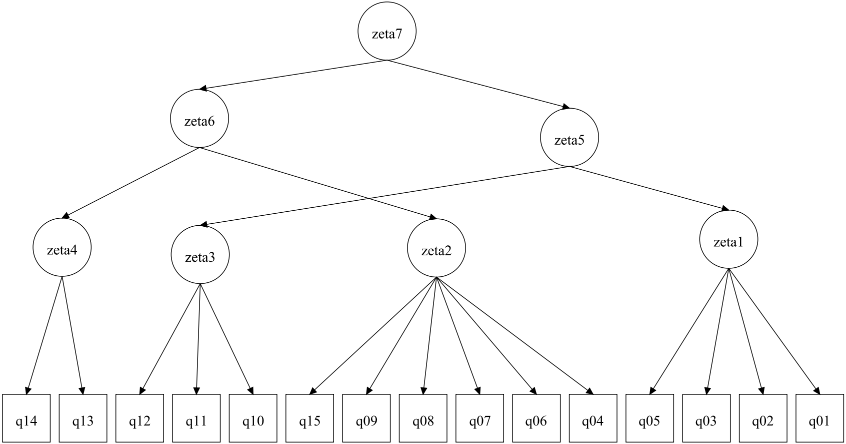
$$\Omega_2 = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{B}\mathbf{\Psi}\mathbf{B}'\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = \omega_5 = .854.$$

Third-order Measurement Model

Now consider more complicated third-order scale.

- Items form a perfect cluster configuration on each of 4 factors
- Two first-order factors are accounted for by a second-order factor.
- Other two first-order factors are accounted for by another second-order factor.
- The two second-order factors are accounted for by a third-order factor.

3rd-Order Measurement Model



3rd-order Multidimensional Reliability

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\Psi}[(\mathbf{I}-\mathbf{B})^{-1}]'\boldsymbol{\Lambda}'\mathbf{1}}{\text{var}(Y)} = 0.954$$

Define subscale reliabilities as before

$$\omega_k = \frac{\mathbf{1}'\tilde{\lambda}_k \boldsymbol{\psi}'_k \tilde{\boldsymbol{\Lambda}}'\mathbf{1}}{\text{var}(Y)}.$$

Then

$$\omega_1 = .028; \quad \omega_2 = .028; \quad \omega_3 = .008; \quad \omega_4 = .024;$$
$$\omega_5 = .008; \quad \omega_6 = .008;$$
$$\omega_7 = .848$$

3rd-order Multidimensional Reliability, cont'd

Again the presence of $(\mathbf{I}-\mathbf{B})^{-1}$ obscures the interpretation of the subscale reliabilities. However, in this case,

$$(\mathbf{I}-\mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2.$$

The layer reliabilities are

$$\Omega_1 = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = \omega_1 + \omega_2 + \omega_3 + \omega_4 = .089;$$

$$\Omega_2 = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{B}\mathbf{\Psi}\mathbf{B}'\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = \omega_5 + \omega_6 = .017.$$

$$\Omega_3 = \frac{\mathbf{1}'\mathbf{\Lambda}\mathbf{B}^2\mathbf{\Psi}\mathbf{B}^{2'}\mathbf{\Lambda}'\mathbf{1}}{\text{var}(Y)} = \omega_7 = .848.$$

Summary

For the extended congeneric measurement model, multidimensional reliability for unit-weighted sum scores is McDonald-Bentler's ω

$$\omega = \frac{\mathbf{1}'\boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\Psi}\left[(\mathbf{I}-\mathbf{B})^{-1}\right]'\boldsymbol{\Lambda}'\mathbf{1}}{\text{var}(Y)}.$$

Letting $\tilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}$, the subscale reliability for factor k $(k = 1, \ldots, m)$ is

$$\omega_k = \frac{\mathbf{1}'\tilde{\boldsymbol{\lambda}}_k\boldsymbol{\psi}'_k\tilde{\boldsymbol{\Lambda}}'\mathbf{1}}{\text{var}(Y)}.$$

The layer reliability for level r $(r = 1, \ldots, s$ and $\mathbf{B}^0 = \mathbf{I})$ of an s-order recursive model is

$$\Omega_r = \frac{\mathbf{1}'\boldsymbol{\Lambda}\mathbf{B}^{r-1}\boldsymbol{\Psi}\left[\mathbf{B}^{r-1}\right]'\boldsymbol{\Lambda}'\mathbf{1}}{\text{var}(Y)}.$$

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- Applies to congeneric and extended congeneric models
- Subscale reliability is general
- Level reliability requires recursive higher-order model.
- Assumes all loadings are non-negative—Negative loadings can yield negative subscale reliability!?!
- There is a parallel development for internal consistency, α.

References

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