Empirical Bayes Derivative Estimates

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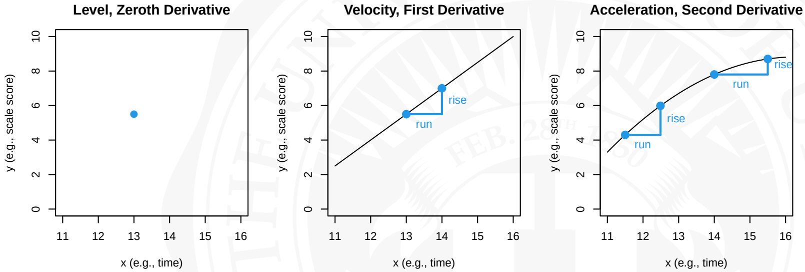
Introduction

- Derivative Estimation, Empirical Bayes Derivative Estimates
- Simulations: Short, Long Time Series
- Substantive Examples

Derivatives



- Zeroth: x, First: dx/dt, Second: d^2x/dt^2

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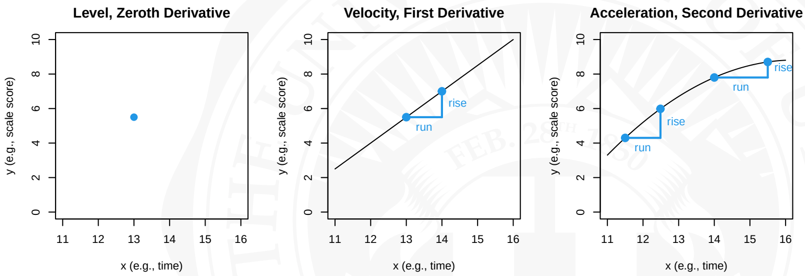
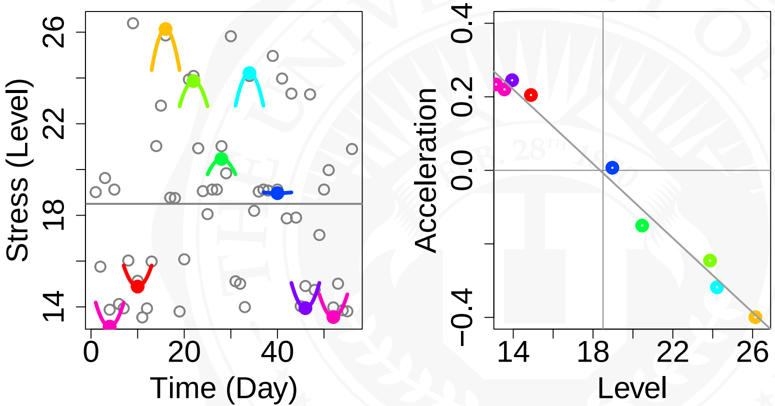


- Zeroth: x, First: dx/dt, Second: d^2x/dt^2
- Relations between derivatives

Differential Equation Modeling



Order-Plus-One

- Difference Scores $(x_t - x_{t-1}, x_{post} - x_{pre})$
  - Zeroth Difference: x_t
  - First Difference: $x_{t+\Delta} - x_t$
  - Second Difference: $(x_{t+\Delta} - x_t) - (x_t - x_{t-\Delta})$

[1] Boker & Nesselroade, 2002

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- Local Linear Approximation[1]
  - Zeroth Derivative: x_t
  - First Derivative: $(x_{t+\Delta} - x_t)/\tau\Delta$
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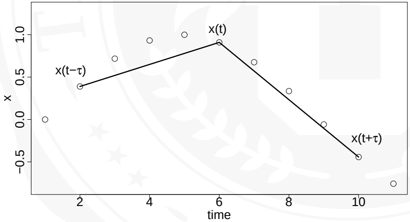
  - Time explicit, τ smoothing



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Regression Methods

- Generalized Local Linear Approximation (GLLA)[2]
  - $X = \{X_t, X_{t+1}, X_{t+2}, X_{t+3}, X_{t+4}\}$
  - $X = \beta_0 + \beta_1(Time) + \beta_2(\frac{1}{2}Time^2) + \varepsilon$
  - Derivatives: β_0, β_1, β_2

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- Generalized Orthogonal Local Derivative Estimates (GOLD)[3]
  - EQ1: $X = \beta_0 + \varepsilon$
  - EQ2: $X = \beta_0 + \beta_1(Time) + \varepsilon$
  - EQ3: $X = \beta_0 + \beta_1(Time) + \beta_2(\frac{1}{2}Time^2) + \varepsilon$
  - Derivatives: β_0 (EQ1), β_1 (EQ2), β_2 (EQ3)

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  - Derivatives: β_0 (EQ1), β_1 (EQ2), β_2 (EQ3)
  - Advantages are the same as GLLA, orthogonal partitioning of variance

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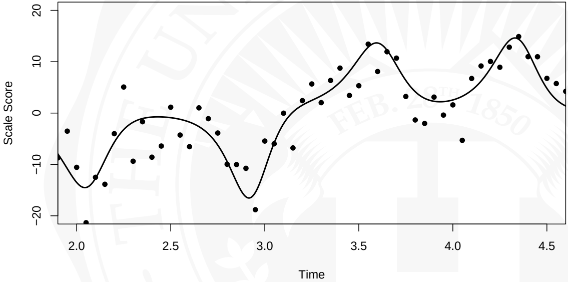
Non-independent Estimation

- Functional Data Analysis[4]
- Repeated observations: complex, but smooth, process



- Polynomial segments (like GLLA), but time-adjacent polynomials the derivatives are constrained to be continuous (to specified derivative)

[4] Ramsay & Silverman, 2002, 2005

Non-independent Estimation

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- Derivatives (Level, Velocity, Acceleration) as random effects

Short Time Series Data Structure

$id(i)$	X_{it}	$Time$	$\frac{1}{2}Time^2$
1	X_{11}	-1.5	1.125
1	X_{12}	-0.5	0.125
1	X_{13}	0.5	0.125
1	X_{14}	1.5	1.125
2	X_{21}	-1.5	1.125
2	X_{22}	-0.5	0.125
2	X_{23}	0.5	0.125
2	X_{24}	1.5	1.125
⋮	⋮	⋮	⋮

Long Time Series Data Structure

id	$Windows(i)$	X_{it}	$Time$	$\frac{1}{2} Time^2$
1	1	X_{11}	-1.5	1.125
1	1	X_{12}	-0.5	0.125
1	1	X_{13}	0.5	0.125
1	1	X_{14}	1.5	1.125
1	2	X_{12}	-1.5	1.125
1	2	X_{13}	-0.5	0.125
1	2	X_{14}	0.5	0.125
1	2	X_{15}	1.5	1.125
1	3	X_{13}	-1.5	1.125
1	3	X_{14}	-0.5	0.125
1	3	X_{15}	0.5	0.125
1	3	X_{16}	1.5	1.125
⋮	⋮	⋮	⋮	⋮

Empirical Bayes Derivative Estimates

$$X_{it} = \beta_{0i} + \beta_{1i}(Time) + \beta_{2i}(\frac{1}{2}Time^2) + \varepsilon_{it}$$
$$\beta_{0i} = \gamma_{00} + u_{0i}$$
$$\beta_{1i} = \gamma_{10} + u_{1i}$$
$$\beta_{2i} = \gamma_{20} + u_{2i}$$

- Multivariate normal distribution of derivatives: β_{0i}, β_{1i}, β_{2i}

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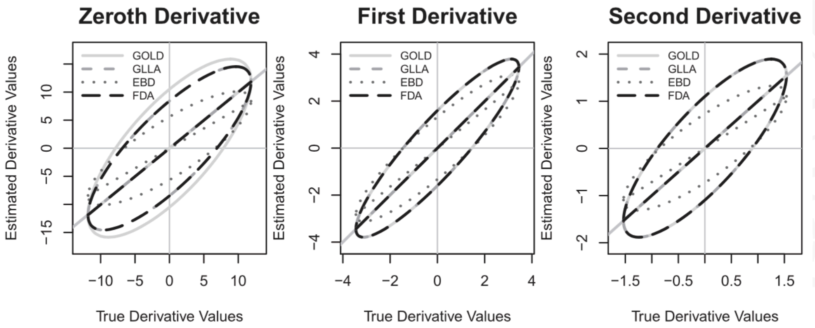
- Multivariate normal distribution of derivatives: β_{0i}, β_{1i}, β_{2i}
- Subsequent estimation of random effects: "Conditional mean/mode" estimates
- Application of Bayes' formula with model estimates as known prior, data for each "group"

Simulations

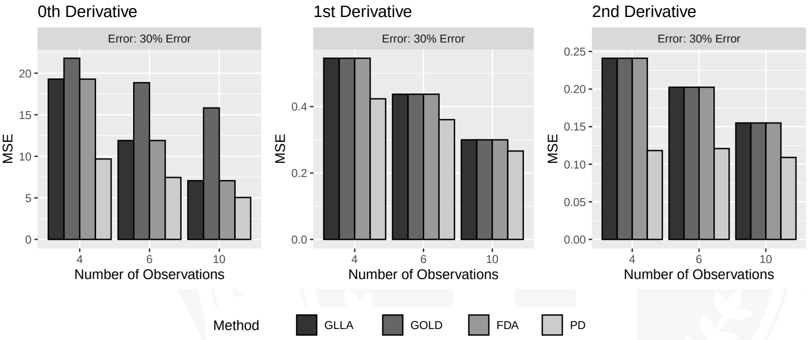
- Two Simulations: Short, Long time series
- GLLA, GOLD, FDA, EBD
- Short Time Series
  - Quadratic time series
  - 4-10 observations per time series
  - 25, 50, 100, 500 time series (i.e., many individuals)
  - Measurement error: 10%, 30%, 50%

Results



- 100 time series, 30% measurement error, 6 observations/time series

Results, MSE



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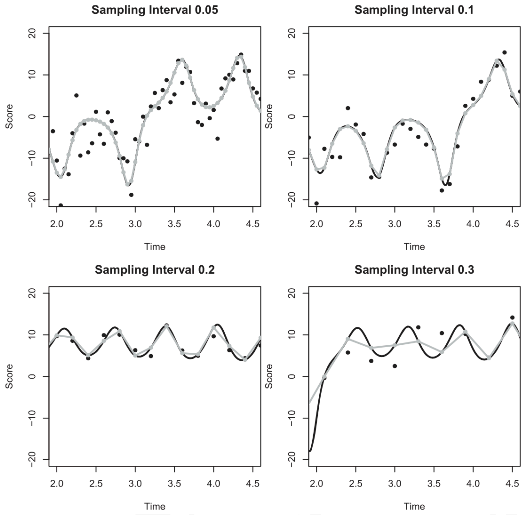
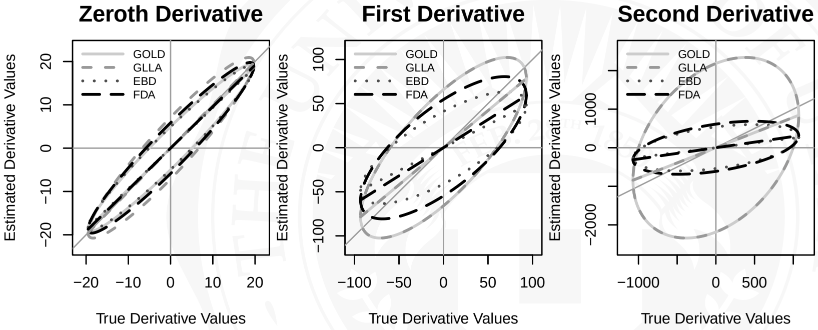
Long Time Series Simulation

- “x” component of Lorenz Attractor
- Varied sampling rate (0.05, 0.10, 0.20, 0.30)
- Embedding 5
- Time series of 25, 50, 100, 500 observations
- Measurement error: 10%, 30%, 50%

Long Time Series, Sampling Rates

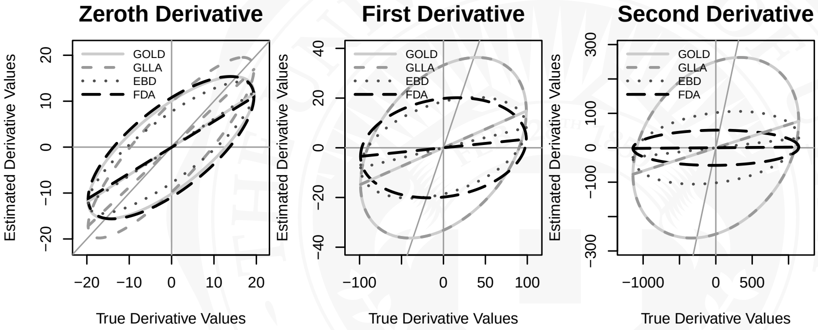


Results: High Fidelity, Sampling Rate 0.05



- Time series length of 100, 30% error

Results: Low Fidelity, Sampling Rate 0.20



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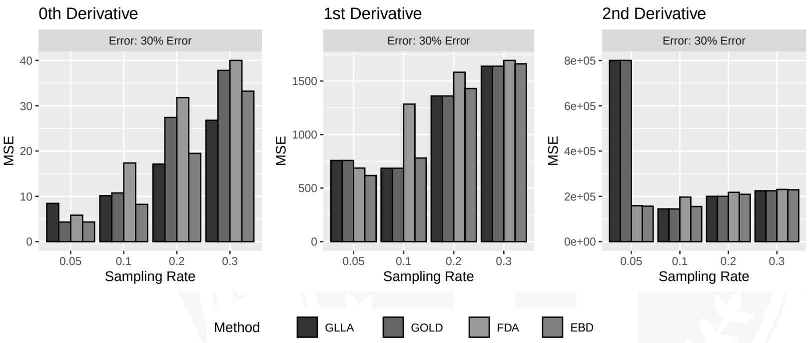

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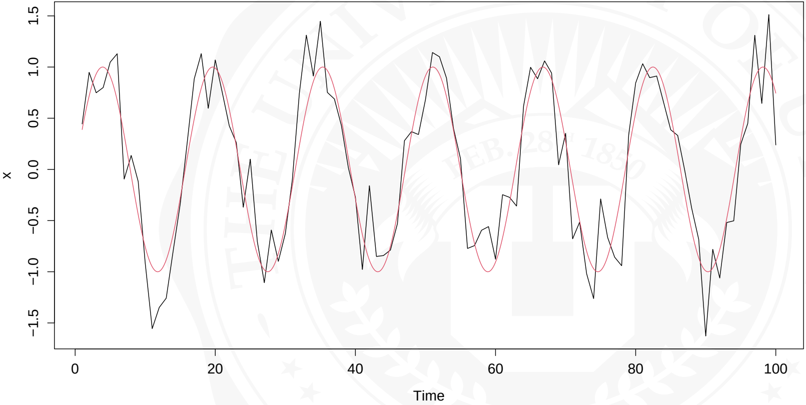
Differencing — GLLA — EBD



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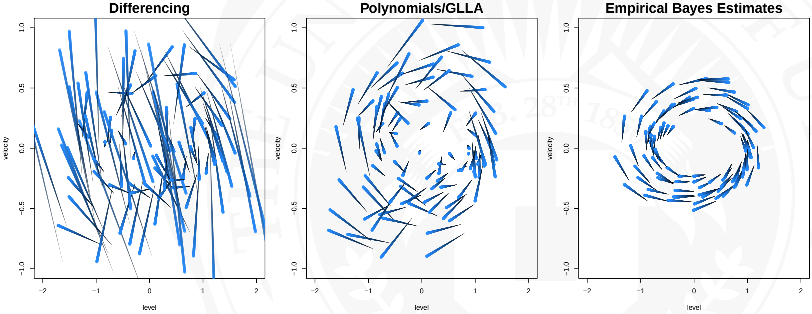

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- Same mean/covariance matrix across all estimates
  - People with Different Dynamics
  - Non-stationary Time Series
- Linear relations between derivatives

Variations

- Multilevel Modeling Advantages

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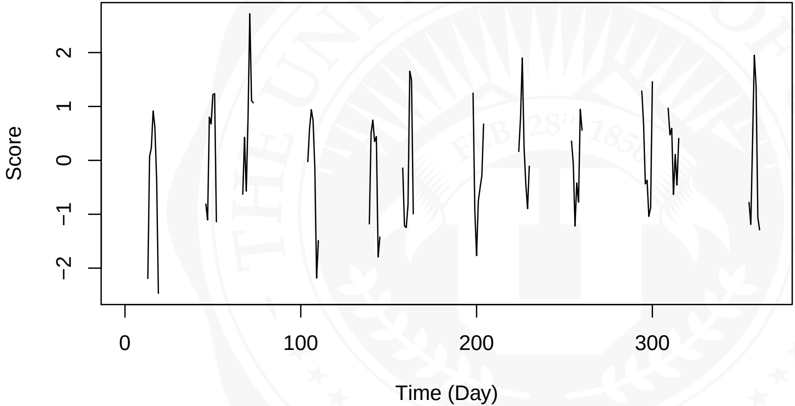
- New intensive sampling schemes

7/30 Sampling Scheme



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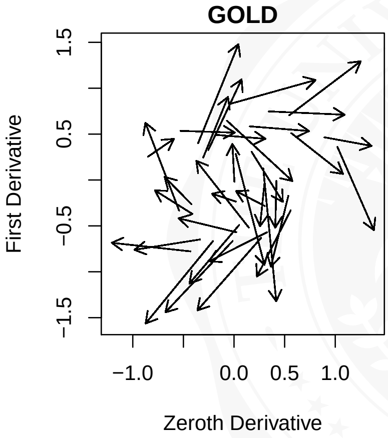

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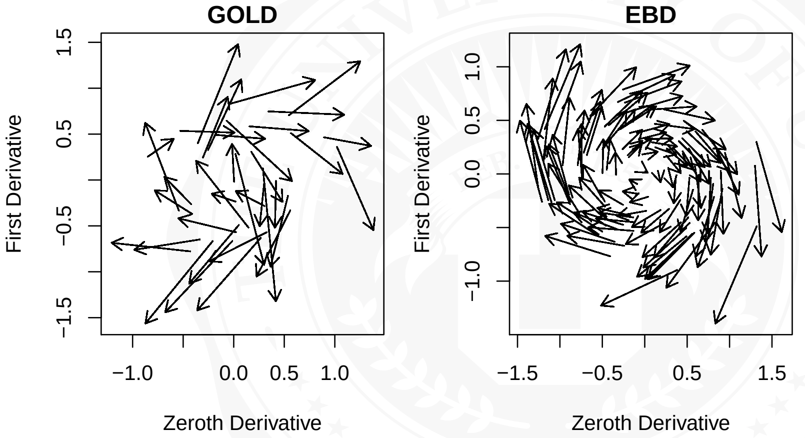
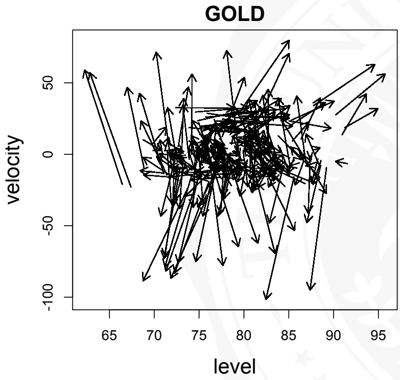


Heart Rate Data



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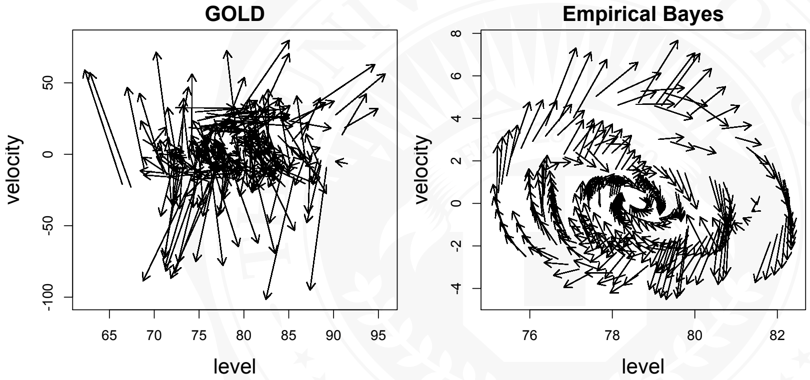
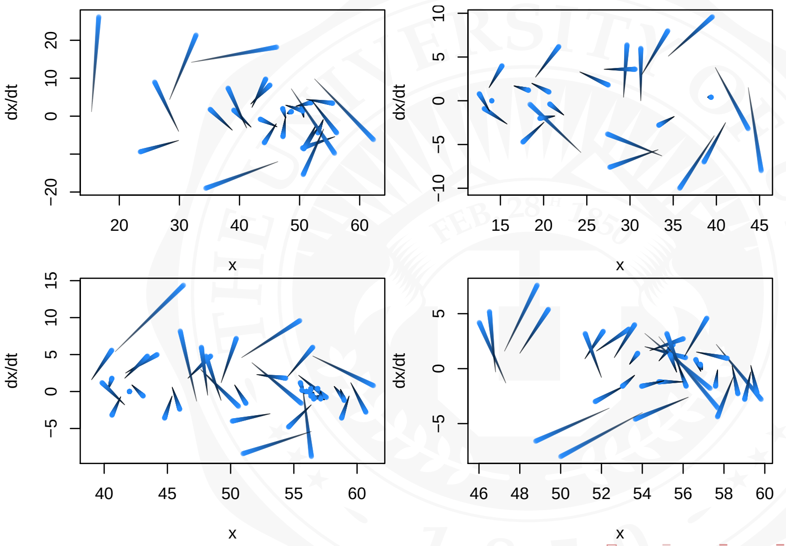


Positive Affect, Daily for 56 Days (GOLD)



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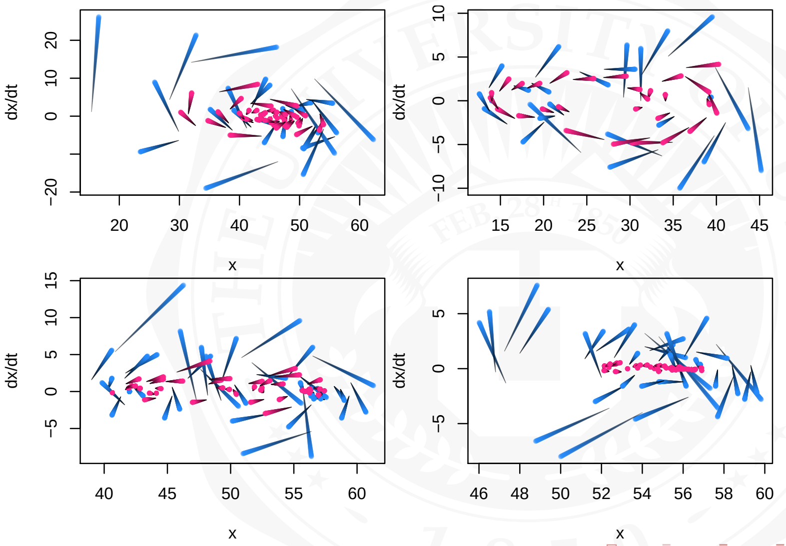

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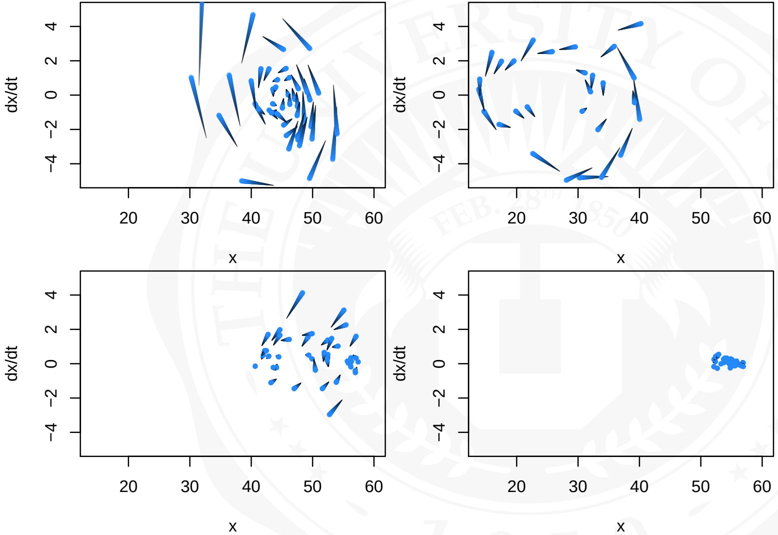

Conclusion

- Derivative Estimation
  - Order-Plus-One (Differences, LLA), Regression Methods (GLLA, GOLD), Non-independent Estimation (FDA)
  - Empirical Bayes Derivative Estimates (Multilevel Models, random effects)
  - Simulations highlight attenuation, greatly reduced variance
  - Generally as efficient or better than GLLA/GOLD/FDA
- Variations
  - Unequal Observation Intervals/Missing Data
  - Generalized Linear Mixed Models (Categorical, Poisson data)
  - Simultaneously Detrend Data
  - New intensive sampling schemes (7/30 for a year)
- Substantive: Heart Rate, Positive Affect

- Paper: Deboeck, P. R. (2020). Empirical Bayes Derivative Estimates. Multivariate Behavioral Research, 55:3, 382–404.

- Substantive State-Space Modeling: Montpetit, M. A., Bergeman, C. S., Deboeck, P. R., Tiberio, S. S. & Boker, S. M. (2010) Resilience-As-Process: Negative Affect, Stress, and Coupled Dynamical Systems. Psychology and Aging, 25 (3), 631–640.
- GLLA: Boker, S. M., Deboeck, P. R., Edler, C. & Keel, P. K. (2009). Generalized Local Linear Approximation of Derivatives from Time Series. In S. Chow, E. Ferrer & F. Hsieh (Eds.), Statistical Methods for Modeling Human Dynamics: An Interdisciplinary Dialogue, pp. 161–178. New York, NY: Taylor & Francis Group.
- GOLD: Deboeck, P. R. (2010). Estimating Dynamical Systems, Derivative Estimation Hints from Sir Ronald A. Fisher. Multivariate Behavioral Research, 43 (4), 725–745.